SHARP INEQUALITIES FOR POLYGAMMA FUNCTIONS

FENG QI AND BAI-NI GUO

ABSTRACT. The main aim of this paper is to prove that the double inequality

$$\frac{(k-1)!}{\left\{x+\left[\frac{(k-1)!}{|\psi^{(k)}(1)|}\right]^{1/k}\right\}^k}+\frac{k!}{x^{k+1}}<|\psi^{(k)}(x)|<\frac{(k-1)!}{\left(x+\frac12\right)^k}+\frac{k!}{x^{k+1}}$$

holds for $x>0$ and $k\in\mathbb{N}$ and that the constants $\left[\frac{(k-1)!}{|\psi^{(k)}(1)|}\right]^{1/k}$ and $\frac12$ are the best possible. In passing, some related inequalities and (logarithmically) complete monotonicity results concerning the gamma, psi and polygamma functions are surveyed.

1. Introduction

1.1. **Completely monotonic functions.** Recall [27, Chapter XIII] and [59, Chapter IV] that a function $f(x)$ is said to be completely monotonic on an interval $I\subseteq\mathbb{R}$ if $f(x)$ has derivatives of all orders on I and

$$0\le(-1)^kf^{(k)}(x)<\infty \tag{1}$$

holds for all $k\ge0$ on I.

The celebrated Bernstein-Widder Theorem [59, p. 161] states that a function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if

$$f(x)=\int_0^\infty e^{-xs}\,\mathrm{d}\mu(s), \tag{2}$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral (2) converges for all $x>0$. This means that a function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform of the measure μ.

The completely monotonic functions have applications in different branches of mathematical sciences. For example, they play some role in combinatorics, numerical and asymptotic analysis, physics, potential theory, and probability theory.

The most important properties of completely monotonic functions can be found in [27, Chapter XIII], [59, Chapter IV] and closely-related references therein.

1.2. **Logarithmically completely monotonic functions.** Recall also [6, 47] that a function f is said to be logarithmically completely monotonic on an interval $I\subseteq\mathbb{R}$ if it has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$0\le(-1)^k[\ln f(x)]^{(k)}<\infty \tag{3}$$

for $k\in\mathbb{N}$ on I.

By looking through "logarithmically completely monotonic function" in the database MathSciNet, it is found that this phrase was first used in [6], but with no a word to explicitly define it. Thereafter, it seems to have been ignored by the mathematical community. In early 2004, this terminology was recovered in [47] and

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it was immediately referenced in [55], the preprint of the paper [54]. A natural question that one may ask is: Whether is this notion trivial or not? In [47, Theorem 4], it was proved that all logarithmically completely monotonic functions are also completely monotonic, but not conversely. This result was formally published when revising [41]. Hereafter, this conclusion and its proofs were dug in [12, 15, 16] and [58] (the preprint of [46]) once and again. Furthermore, in the paper [12], the logarithmically completely monotonic functions on $(0,\infty)$ were characterized as the infinitely divisible completely monotonic functions studied in [22] and all Stieltjes transforms were proved to be logarithmically completely monotonic on $(0,\infty)$. For more information, please refer to [12].

1.3. **The gamma and polygamma functions.** It is well-known that the classical Euler gamma function $\Gamma(x)$ may be defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t. \tag{4}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ are called the polygamma functions. It is common knowledge that these functions are fundamental and important and that they have much extensive applications in mathematical sciences.

1.4. **The first kind of inequalities for the psi and polygamma functions.** In [23, Theorem 2.1], [38, Lemma 1.3] and [39, Lemma 3], the function $\psi(x)-\ln x+\frac{\alpha}{x}$ was proved to be completely monotonic on $(0,\infty)$ if and only if $\alpha\ge1$, so is its negative if and only if $\alpha\le\frac12$. In [13, Theorem 2] and [28, Theorem 2.1], the function $\frac{e^x\Gamma(x)}{x^{x-\alpha}}$ was proved to be logarithmically completely monotonic on $(0,\infty)$ if and only if $\alpha\ge1$, so is its reciprocal if and only if $\alpha\le\frac12$. From these, the following double inequalities were derived and employed in [17, 36, 40, 44, 45, 48, 49, 50, 52, 56, 57]: For $x\in(0,\infty)$ and $k\in\mathbb{N}$, we have

$$\ln x-\frac1x<\psi(x)<\ln x-\frac1{2x} \tag{5}$$

and

$$\frac{(k-1)!}{x^k}+\frac{k!}{2x^{k+1}}<\bigl|\psi^{(k)}(x)\bigr|<\frac{(k-1)!}{x^k}+\frac{k!}{x^{k+1}}. \tag{6}$$

In [3, Theorem 9], it was proved that if $k\ge1$ and $n\ge0$ are integers then

$$S_k(2n;x)<\bigl|\psi^{(k)}(x)\bigr|<S_k(2n+1;x) \tag{7}$$

holds for $x>0$, where

$$S_k(p;x)=\frac{(k-1)!}{x^k}+\frac{k!}{2x^{k+1}}+\sum_{i=1}^pB_{2i}\left[\prod_{j=1}^{k-1}(2i+j)\right]\frac1{x^{2i+k}} \tag{8}$$

with the usual convention that an empty sum is nil and B_i for $i\ge0$ are Bernoulli numbers defined by

$$\frac{t}{e^t-1}=\sum_{i=0}^\infty B_i\frac{t^i}{i!}=1-\frac x2+\sum_{j=1}^\infty B_{2j}\frac{x^{2j}}{(2j)!},\quad |x|<2\pi. \tag{9}$$

In [2], among other things, the following double inequalities were procured: For $x>\frac12$, we have

$$\sum_{k=1}^{2N+1}\frac{B_{2k}\bigl(\frac12\bigr)}{2k\bigl(x-\frac12\bigr)^{2k}}<\ln\biggl(x-\frac12\biggr)-\psi(x)<\sum_{k=1}^{2N}\frac{B_{2k}\bigl(\frac12\bigr)}{2k\bigl(x-\frac12\bigr)^{2k}} \tag{10}$$

and

$$\frac{(n-1)!}{\left(x-\frac{1}{2}\right)^n}+\sum_{k=1}^{2N+1}\frac{(n+2k-1)!B_{2k}\left(\frac{1}{2}\right)}{(2k)!\left(x-\frac{1}{2}\right)^{n+2k}}<\left|\psi^{(n)}(x)\right| \\ <\frac{(n-1)!}{\left(x-\frac{1}{2}\right)^n}+\sum_{k=1}^{2N}\frac{(n+2k-1)!B_{2k}\left(\frac{1}{2}\right)}{(2k)!\left(x-\frac{1}{2}\right)^{n+2k}}, \quad (11)$$

where $n\ge 1$, $N\ge 0$, an empty sum is understood to be nil, and

$$B_k\left(\frac{1}{2}\right)=\left(\frac{1}{2^{k-1}}-1\right)B_k, \quad k\ge 0. \quad (12)$$

When replacing $2N$ by $2N-1$, inequalities (10) and (11) are reversed. In particular, for $n=1$ and $N=0$,

$$\frac{1}{x-\frac{1}{2}}-\frac{1}{12\left(x-\frac{1}{2}\right)^2}<\psi'(x)<\frac{1}{x-\frac{1}{2}}, \quad x>\frac{1}{2}. \quad (13)$$

It is obvious that if taking $x\to\left(\frac{1}{2}\right)^+$ the lower and upper bounds in (11) tend to $-\infty$ and ∞ respectively, but the middle term tends to a limited constant. This implies that inequalities in (10) and (11), including (13), may be not ideal.

It is noted that the inequality (7) was deduced from [3, Theorem 8] which states that the functions

$$F_n(x)=\ln\Gamma(x)-\left(x-\frac{1}{2}\right)\ln x+x-\frac{1}{2}\ln(2\pi)-\frac{1}{2}\sum_{j=1}^{2n}\frac{B_{2j}}{2j(2j-1)x^{2j-1}} \quad (14)$$

and

$$G_n(x)=-\ln\Gamma(x)+\left(x-\frac{1}{2}\right)\ln x-x+\frac{1}{2}\ln(2\pi)+\frac{1}{2}\sum_{j=1}^{2n+1}\frac{B_{2j}}{2j(2j-1)x^{2j-1}} \quad (15)$$

are completely monotonic on $(0,\infty)$.

In [26, Theorem 1], the convexity of the functions $F_n(x)$ and $G_n(x)$ were presented alternatively.

Stimulated by [42], the complete monotonicity of $F_n(x)$ and $G_n(x)$ were simply verified in [25, Theorem 2] again.

1.5. **The second kind of inequalities for the psi and polygamma functions.** In [19, Theorem 1], the function

$$g_{\alpha,\beta}(x)=\left[\frac{e^x\Gamma(x+1)}{(x+\beta)^{x+\beta}}\right]^\alpha \quad (16)$$

for real numbers $\alpha\ne 0$ and β was shown to be logarithmically completely monotonic with respect to $x\in(\max\{0,-\beta\},\infty)$ if and only if either $\alpha>0$ and $\beta\ge 1$ or $\alpha<0$ and $\beta\le\frac{1}{2}$. As a result, the following double inequalities (17) and (18) were deduced in [49, Lemma 2] and used in [49, Lemma 3]: For $x\in(0,\infty)$ and $k\in\mathbb{N}$, we have

$$\ln\left(x+\frac{1}{2}\right)-\frac{1}{x}<\psi(x)<\ln(x+1)-\frac{1}{x} \quad (17)$$

and

$$\frac{(k-1)!}{(x+1)^k}+\frac{k!}{x^{k+1}}<\left|\psi^{(k)}(x)\right|<\frac{(k-1)!}{\left(x+\frac{1}{2}\right)^k}+\frac{k!}{x^{k+1}}. \quad (18)$$

It is clear that the left-hand side inequality in (17) and the right-hand side inequality in (18) are better than the left-hand side inequality in (5) and the right-hand side inequality in (6). It is also easy to see that the right-hand side inequality in (17) and the left-hand side inequality in (18) are more exact than the right-hand

side inequality in (5) and the left-hand side inequality in (6) when $x > 0$ is close enough to 0, but not when $x > 0$ is large enough.

For more information on further investigation of functions similar to (16), please refer to the research papers [18, 20, 21], the expository article [32] and related references therein.

1.6. **A sharp inequality for the psi function and related results.** In [8, Lemma 1.7] and [51, Theorem 1], it was proved that the double inequality

$$\ln\left(x+\frac{1}{2}\right)-\frac{1}{x}<\psi(x)<\ln(x+e^{-\gamma})-\frac{1}{x} \tag{19}$$

holds on $(0,\infty)$ and the scalars $\frac{1}{2}$ and $e^{-\gamma}=0.56\cdots$ in (19) are the best possible.

It is clear that the inequality (19) refines and sharpens (17). The inequality (19) has relations with (5) as (17) does.

More strongly, the function

$$Q(x)=e^{\psi(x+1)}-x \tag{20}$$

was proved in [51, Theorem 2] to be strictly decreasing and convex on $(-1,\infty)$ with $\lim_{x\to\infty}Q(x)=\frac{1}{2}$. The basic tools of the proofs in [51] include

$$\psi'(x)e^{\psi(x)}<1, \quad x>0 \tag{21}$$

and

$$[\psi'(x)]^2+\psi''(x)>0, \quad x>0. \tag{22}$$

Among other things, the monotonicity and convexity of the function (20) were also derived in [14, Corollary 2 and Corollary 3]: For all $t>0$, the function $\exp\{\psi(x+t)\}-x$ is decreasing with respect to $x\in[0,\infty)$; For all $t>0$, the digamma function can be written in a way:

$$\psi(x+t)=\ln(x+\delta(x)), \quad x>0,$$

where δ is decreasing convex function which maps $[0,\infty)$ onto $\left[e^{\psi(t)},t-\frac{1}{2}\right)$.

The one-sided inequality (21) was deduced in [14, Corollary 2] and recovered in [10, Lemma 1.1] and [11, Lemma 1.1].

The sharp double inequality (19) is the special case $t=1$ of the following double inequality obtained in [14, Corollary 3]: For all $x>0$ and $t>0$, it holds that

$$\ln\left(x+\frac{2t-1}{2}\right)<\psi(x+t)<\ln(x+\exp(\psi(t))). \tag{23}$$

It is worthwhile to remark that the left-hand side inequality in (23) for $x+t\le\frac{1}{2}$ is meaningless.

Replacing x by $x+t$ in (19) yields

$$\ln\left(x+t+\frac{1}{2}\right)-\frac{1}{x+t}<\psi(x+t)<\ln(x+t+e^{-\gamma})-\frac{1}{x+t} \tag{24}$$

for all $x>0$ and $t>0$. The left-hand side inequality in (24) extends and refines the corresponding one in (23) and their right-hand side inequalities do not contain each other.

For information about the history and backgrounds of the function (20) and inequalities (21) and (23), please refer to the expository papers [32, 34] and lots of references therein.

The inequality (22) was first obtained in the proof of [4, p. 208, Theorem 4.8] and recovered in [7, Theorem 2.1], [10, Lemma 1.1] and [11, Lemma 1.1].

In [9, Remark 1.3], it was pointed out that the inequality (22) is the special case $n=1$ of [9, Lemma 1.2] which reads

$$(-1)^n\psi^{(n+1)}(x)<\frac{n}{\sqrt[n]{(n-1)!}}\big[(-1)^{n-1}\psi^{(n)}(x)\big]^{1+1/n} \tag{25}$$

for $x>0$ and $n\in\mathbb{N}$. This inequality can be restated more meaningfully as

$$\sqrt[n+1]{\frac{\big|\psi^{(n+1)}(x)\big|}{n!}}<\sqrt[n]{\frac{\big|\psi^{(n)}(x)\big|}{(n-1)!}}. \tag{26}$$

In [5, Lemma 4.6], the inequality (22) was generalized to the q-analogue.

In [31, 37], the preprints of [43, 53], the divided difference

$$\Delta_{s,t}(x)=\begin{cases}\left[\dfrac{\psi(x+t)-\psi(x+s)}{t-s}\right]^2+\dfrac{\psi'(x+t)-\psi'(x+s)}{t-s}, & s\ne t\\ [\psi'(x+s)]^2+\psi''(x+s), & s=t\end{cases} \tag{27}$$

for $|t-s|<1$ and $-\Delta_{s,t}(x)$ for $|t-s|>1$ were proved to be completely monotonic with respect to $x\in(-\min\{s,t\},\infty)$. In particular, the function $[\psi'(x)]^2+\psi''(x)$ appearing in (22) is completely monotonic on $(0,\infty)$.

For $m,n\in\mathbb{N}$, let

$$f_{m,n}(x)=\psi^{(n)}(x)+\big[\psi^{(m)}(x)\big]^2,\quad x>0. \tag{28}$$

In [48], it was revealed that the functions $f_{1,2}(x)$ and $f_{m,2n-1}(x)$ are completely monotonic on $(0,\infty)$, but the functions $f_{m,2n}(x)$ for $(m,n)\ne(1,1)$ are not monotonic and does not keep the same sign on $(0,\infty)$. This means that $f_{1,2}(x)$ is the only nontrivial completely monotonic function on $(0,\infty)$ among all functions $f_{m,n}(x)$ for $m,n\in\mathbb{N}$.

In [50], the function

$$\Delta_\lambda(x)=[\psi'(x)]^2+\lambda\psi''(x) \tag{29}$$

was shown to be completely monotonic on $(0,\infty)$ if and only if $\lambda\le1$.

For real numbers s, t, $\alpha=\min\{s,t\}$ and λ, define

$$\Delta_{s,t;\lambda}(x)=\begin{cases}\left[\dfrac{\psi(x+t)-\psi(x+s)}{t-s}\right]^2+\lambda\dfrac{\psi'(x+t)-\psi'(x+s)}{t-s}, & s\ne t\\ [\psi'(x+s)]^2+\lambda\psi''(x+s), & s=t\end{cases} \tag{30}$$

with respect to $x\in(-\alpha,\infty)$. In [36], the following complete monotonicity were established:

(1) For $0<|t-s|<1$,
   (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha,\infty)$ if and only if $\lambda\le1$,
   (b) so is the function $-\Delta_{s,t;\lambda}(x)$ if and only if $\lambda\ge\frac{1}{|t-s|}$;
(2) For $|t-s|>1$,
   (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha,\infty)$ if and only if $\lambda\le\frac{1}{|t-s|}$,
   (b) so is the function $-\Delta_{s,t;\lambda}(x)$ if and only if $\lambda\ge1$;
(3) For $s=t$, the function $\Delta_{s,s;\lambda}(x)$ is completely monotonic on $(-s,\infty)$ if and only if $\lambda\le1$;
(4) For $|t-s|=1$,
   (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda<1$,
   (b) so is the function $-\Delta_{s,t;\lambda}(x)$ if and only if $\lambda>1$,
   (c) and $\Delta_{s,t;1}(x)\equiv0$.

These results generalize the claim in the proof of [24]. For detailed information, see related texts remarked in the expository article [33].

In [9, Remark 2.3], it was pointed out that the inequality

$$\psi''(x)+\left[\psi'\left(x+\frac{1}{2}\right)\right]^2<0 \tag{31}$$

for $x>0$ is a direct consequence of [9, Theorem 2.2]: For $x>0$, $1\le k\le n-1$ and $n\in\mathbb{N}$, we have

$$\begin{aligned}(n-1)!\left[\frac{\psi^{(k)}\left(x+\frac{1}{2}\right)}{(-1)^{k-1}(k-1)!}\right]^{n/k}&<(-1)^{n+1}\psi^{(n)}(x)\\&<(n-1)!\left[\frac{\psi^{(k)}(x)}{(-1)^{k-1}(k-1)!}\right]^{n/k}\end{aligned} \tag{32}$$

which can be rewritten as

$$\sqrt[k]{\frac{\left|\psi^{(k)}\left(x+\frac{1}{2}\right)\right|}{(k-1)!}}<\sqrt[n]{\frac{\left|\psi^{(n)}(x)\right|}{(n-1)!}}<\sqrt[k]{\frac{\left|\psi^{(k)}(x)\right|}{(k-1)!}}. \tag{33}$$

In [50], the inequality (31) was generalized to the complete monotonicity: For real number $\alpha\in\mathbb{R}$ and $x>-\min\{0,\alpha\}$,

(1) the function $\psi''(x)+[\psi'(x+\alpha)]^2$ is completely monotonic if and only if $\alpha\le0$;
(2) the function $-\left\{\psi''(x)+[\psi'(x+\alpha)]^2\right\}$ is completely monotonic if

$$\alpha\ge\sup_{x\in(0,\infty)}\frac{x}{\phi^{-1}\left([2(x+1)^2-1]e^{2x}\right)}, \tag{34}$$

where ϕ^{-1} denotes the inverse function of $\phi(x)=x\coth x$ on $(0,\infty)$.

In passing, it is noted that the results demonstrated in [29, 30, 35, 40] have very close relations with the above mentioned conclusions.

1.7. **Main results of this paper.** The main aim of this paper is to sharpen the double inequality (18) and to generalize the sharp inequality (19) to the cases for polygamma functions.

The main result of this paper may be stated as the following theorem.

Theorem 1. *For $x>0$ and $k\in\mathbb{N}$, the double inequality*

$$\frac{(k-1)!}{\left\{x+\left[\frac{(k-1)!}{|\psi^{(k)}(1)|}\right]^{1/k}\right\}^k}+\frac{k!}{x^{k+1}}<\left|\psi^{(k)}(x)\right|<\frac{(k-1)!}{\left(x+\frac{1}{2}\right)^k}+\frac{k!}{x^{k+1}} \tag{35}$$

holds and the constants $\left[\frac{(k-1)!}{|\psi^{(k)}(1)|}\right]^{1/k}$ and $\frac{1}{2}$ in (35) are the best possible.

As direct consequences of Theorem 1, the following corollaries may be derived.

Corollary 1. *For $x>0$ and $k\in\mathbb{N}$, the double inequality*

$$\frac{(k-1)!}{\left\{x+\left[\frac{(k-1)!}{|\psi^{(k)}(1)|}\right]^{1/k}\right\}^k}<\left|\psi^{(k)}(x+1)\right|<\frac{(k-1)!}{\left(x+\frac{1}{2}\right)^k} \tag{36}$$

is valid and the scalars $\left[\frac{(k-1)!}{|\psi^{(k)}(1)|}\right]^{1/k}$ and $\frac{1}{2}$ in (36) are the best possible.

Corollary 2. *Under the usual convention that an empty sum is understood to be nil, the double inequalities*

$$k!\sum_{i=1}^{m}\frac{1}{(x+i-1)^{k+1}}+\frac{(k-1)!}{\Bigl\{x+m-1+\Bigl[\frac{(k-1)!}{|\psi^{(k)}(1)|}\Bigr]^{1/k}\Bigr\}^k}<\bigl|\psi^{(k)}(x)\bigr| < k!\sum_{i=1}^{m}\frac{1}{(x+i-1)^{k+1}}+\frac{(k-1)!}{\bigl(x+m-\frac{1}{2}\bigr)^k} \tag{37}$$

and

$$\frac{(k-1)!}{\Bigl\{x+\Bigl[\frac{(k-1)!}{|\psi^{(k)}(1)|}\Bigr]^{1/k}\Bigr\}^k}-k!\sum_{i=1}^{m-1}\frac{1}{(x+i)^{k+1}}<\bigl|\psi^{(k)}(x+m)\bigr| < \frac{(k-1)!}{\bigl(x+\frac{1}{2}\bigr)^k}-k!\sum_{i=1}^{m-1}\frac{1}{(x+i)^{k+1}} \tag{38}$$

hold for $x>0$ and $k,m\in\mathbb{N}$. Meanwhile, the quantities $\Bigl[\frac{(k-1)!}{|\psi^{(k)}(1)|}\Bigr]^{1/k}$ and $\frac{1}{2}$ in inequalities (37) and (38) are the best possible.

Remark 1. When approximating the psi function $\psi(x)$ and polygamma functions $\psi^{(k)}(x)$ for $k\in\mathbb{N}$, the double inequalities (19) and (35) are more accurate than (5) and (6) as long as x is enough close to 0 from the right-hand side. For example, the right-hand side inequality in (18) and (35) has been applied in the proof of [49, Lemma 3] to prove that the inequality

$$\frac{1+2t}{2t^2}\biggl[\ln\Gamma\biggl(\frac{t}{1+2t}\biggr)-\ln\Gamma(t)\biggr]<1-\psi(t) \tag{39}$$

is valid for $t>0$.

2. Proofs of Theorem 1 and corollaries

Now we are in a position to prove Theorem 1 and the above corollaries.

Proof of Theorem 1. For $x>0$ and $k\in\mathbb{N}$, let

$$h_k(x)=\biggl[\frac{(k-1)!}{\bigl|\psi^{(k)}(x)\bigr|-\frac{k!}{x^{k+1}}}\biggr]^{1/k}-x. \tag{40}$$

Using the right-hand side inequality in (7) for $n\ge0$ yields

$$\begin{aligned}
h_k(x)&>\biggl[\frac{(k-1)!}{S_k(2n+1;x)-\frac{k!}{x^{k+1}}}\biggr]^{1/k}-x\\
&=\Biggl\{\frac{(k-1)!}{\frac{(k-1)!}{x^k}-\frac{k!}{2x^{k+1}}+\sum_{i=1}^{2n+1}B_{2i}\bigl[\prod_{j=1}^{k-1}(2i+j)\bigr]\frac{1}{x^{2i+k}}}\Biggr\}^{1/k}-x\\
&=x\Biggl\{\biggl[\frac{1}{1-\frac{k}{2x}+\sum_{i=1}^{2n+1}\frac{B_{2i}}{(k-1)!}\bigl[\prod_{j=1}^{k-1}(2i+j)\bigr]\frac{1}{x^{2i}}}\biggr]^{1/k}-1\Biggr\}\\
&=\frac{1}{u}\Biggl\{\biggl[\frac{1}{1-\frac{k}{2}u+\sum_{i=1}^{2n+1}\frac{B_{2i}}{(k-1)!}\bigl[\prod_{j=1}^{k-1}(2i+j)\bigr]u^{2i}}\biggr]^{1/k}-1\Biggr\},\\
&\to\frac{1}{2}\quad\text{as } u\to0^+,\text{ or say, } x\to\infty.
\end{aligned}$$

Similarly, making use of the left-hand side inequality in (7) for $n \ge 0$ results in

$$\begin{aligned}
h_k(x) &< \left[\frac{(k-1)!}{S_k(2n;x)-\frac{k!}{x^{k+1}}}\right]^{1/k} - x \\
&= \left\{\frac{(k-1)!}{\frac{(k-1)!}{x^k}-\frac{k!}{2x^{k+1}}+\sum_{i=1}^{2n}B_{2i}\left[\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i+k}}}\right\}^{1/k} - x \\
&= x\left\{\left[\frac{1}{1-\frac{k}{2x}+\sum_{i=1}^{2n}\frac{B_{2i}}{(k-1)!}\left[\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i}}}\right]^{1/k} - 1\right\} \\
&= \frac{1}{u}\left\{\left[\frac{1}{1-\frac{k}{2}u+\sum_{i=1}^{2n}\frac{B_{2i}}{(k-1)!}\left[\prod_{j=1}^{k-1}(2i+j)\right]u^{2i}}\right]^{1/k} - 1\right\}, \\
&\to \frac{1}{2} \quad \text{as } u \to 0^+, \text{ or say, } x \to \infty.
\end{aligned}$$

In a word, it follows that

$$\lim_{x\to\infty} h_k(x) = \frac{1}{2}. \tag{41}$$

By the well-known recurrence formula [1, p. 260, 6.4.6]

$$\psi^{(n-1)}(x+1) = \psi^{(n-1)}(x) + \frac{(-1)^{n-1}(n-1)!}{x^n} \tag{42}$$

and the integral representation [1, p. 260, 6.4.1]

$$\psi^{(k)}(x) = (-1)^{k+1}\int_0^\infty \frac{t^k}{1-e^{-t}}e^{-xt}\,\mathrm{d}t \tag{43}$$

for $x > 0$ and $n \in \mathbb{N}$, we have

$$\lim_{x\to0^+} h_k(x) = \lim_{x\to0^+}\left[\frac{(k-1)!}{\left|\psi^{(k)}(x+1)\right|}\right]^{1/k} = \left[\frac{(k-1)!}{\left|\psi^{(k)}(1)\right|}\right]^{1/k}. \tag{44}$$

By virtue of (42) and (43), straightforward computation yields

$$\begin{aligned}
h_k'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\left[\frac{(k-1)!}{(-1)^{k+1}\psi^{(k)}(x+1)}\right]^{1/k} - x\right\} \\
&= -\frac{\psi^{(k+1)}(x+1)}{k\psi^{(k)}(x+1)}\left[\frac{(k-1)!}{(-1)^{k+1}\psi^{(k)}(x+1)}\right]^{1/k} - 1 \\
&= \frac{\left|\psi^{(k+1)}(x+1)\right|}{k!}\left[\frac{(k-1)!}{\left|\psi^{(k)}(x+1)\right|}\right]^{1+1/k} - 1 \\
&= \left[\sqrt[k+1]{\frac{\left|\psi^{(k+1)}(x+1)\right|}{k!}}\sqrt[k]{\frac{(k-1)!}{\left|\psi^{(k)}(x+1)\right|}}\right]^{k+1} - 1.
\end{aligned}$$

By virtue of the inequality (26), it follows that $h_k'(x) < 0$ on $(0,\infty)$, which means that the functions $h_k(x)$ for $k \in \mathbb{N}$ are strictly decreasing on $(0,\infty)$.

In conclusion, from a combination of the decreasing monotonicity of $h_k(x)$ with the limits (41) and (44), Theorem 1 follows immediately. □

Proof of Corollary 1. This follows from

$$\left|\psi^{(k)}(x)\right| - \frac{k!}{x^{k+1}} = \left|\psi^{(k)}(x+1)\right|,$$

an equivalence of (42), for $k \in \mathbb{N}$ and $x > 0$. □

Proof of Corollary 2. Utilizing the identity (42) and the integral express (43) shows

$$\left|\psi^{(k)}(x+m)\right| = \left|\psi^{(k)}(x)\right| - \sum_{i=1}^{m} \frac{k!}{(x+i-1)^{k+1}}$$

for $k, m \in \mathbb{N}$ and $x > 0$. Combining this with (35) and the case $x+m$ of (35) respectively leads to the inequalities (37) and (38). Corollary 2 is proved. □

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(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: http://qifeng618.spaces.live.com

(B.-N. Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E-mail address: bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com
URL: http://guobaini.spaces.live.com